All Along the Ring Tower

Algebraic Structures for Fun and Profit

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joint work w/
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Rings in Lattice-Based Cryptography

It is typical in lattice-based cryptography to use matrices with coefficients in $\mathbb{Z}_q[x]/(x^d+1)$ rather than $\mathbb{Z}_q$:

1. Communication costs typically go $O(d^2) \Rightarrow O(d)$
2. Computation costs typically go $O(d^2) \Rightarrow O(d \log d)$

But in some situations this additional structure seems ineffective:

1. Matrix decomposition (Cholesky, Gram-Schmidt, etc.)
2. Solving equations in a ring which is not a field (e.g. $\mathbb{Z}[x]/(x^d+1)$)

Algorithms can take time up to $\Theta(d^2)$ or $\Theta(d^3)$.

The State of Affairs

What naïve solutions do:

1. View $\mathbb{Q}[x]/(x^d + 1)$ as either a $\mathbb{Q}$-linear space of dimension d, an extension field of $\mathbb{Q}$ of degree d, etc.
2. This ignores the rich structure of cyclotomic rings and fields.

What happens when we open the black box?

Cyclotomic Fields - Opening the Black Box

For d a power-of-two, we note:

- $\mathcal{Q}_d = \mathbb{Q}[x]/(x^d + 1)$ the d-th *cyclotomic field*
- $\mathcal{Z}_d = \mathbb{Z}[x]/(x^d + 1)$ the d-th *cyclotomic ring*

We have this tower of fields:

$$\mathbb{Q} \subsetneq \mathcal{Q}_2 \subsetneq \cdots \subseteq \mathcal{Q}_{d/2} \subsetneq \mathcal{Q}_d$$

As well as this chain of isomorphisms:

$$\mathbb{Q}^d \cong (\mathcal{Q}_2)^{d/2} \cong \ldots \cong (\mathcal{Q}_{d/2})^2 \cong \mathcal{Q}_d$$

At a high level:

- The **field norm** and **field trace** allows to move in the tower of fields
- **Ring isomorphisms** allow us to move in the chain of ring isomorphisms

Traces and Norms in Cyclotomic Fields

Definition: For a (finite) field extension L/K:

- The field trace is:

$$\mathrm{Tr}_{L/K} : \quad L \to K, \qquad f \mapsto \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma(f)$$

- The field norm is:

$$\mathrm{N}_{L/K} : \quad L \to K, \qquad f \mapsto \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(f)$$

Concretely: if $f(x) = f_e(x^2) + x \cdot f_o(x^2) \in \mathcal{Q}_d$, then $f^\times(x) = f(-x)$ and:

- $\mathrm{Tr}_{\mathcal{Q}_d/\mathcal{Q}_{d/2}}(f) = f + f^\times = 2 \cdot f_e(x^2)$
- $\mathrm{N}_{\mathcal{Q}_d/\mathcal{Q}_{d/2}}(f) = f \cdot f^\times = f_e^2(x^2) - x^2 f_o^2(x^2)$

Composition properties:

- $\mathrm{Tr}_{L/K} \circ \mathrm{Tr}_{M/L} = \mathrm{Tr}_{M/K}$
- $\mathrm{N}_{L/K} \circ \mathrm{N}_{M/L} = \mathrm{N}_{M/K}$

Homomorphic properties:

- $\mathrm{Tr}_{L/K}(a+b) = \mathrm{Tr}_{L/K}(a) + \mathrm{Tr}_{L/K}(b)$
- $\mathrm{N}_{L/K}(a \cdot b) = \mathrm{N}_{L/K}(a) \cdot \mathrm{N}_{L/K}(b)$

Problem 1 - Completing NTRU Bases

NTRU Lattices:

- Prevalent in lattice-based crypto
- Public key is $\mathbf{A} = \left[\begin{array}{c|c} 1 & h \end{array}\right]$, for $h = g \times f^{-1} \bmod (\varphi, q)$.
- Private key is $\mathbf{B}$ such that $\mathbf{B} \times \mathbf{A}^t = 0 \bmod (\varphi, q)$

Some schemes only require a partial trapdoor $\mathbf{B} = \left[\begin{array}{c|c} g & -f \end{array}\right]$:

- Fiat-Shamir [ZCHW17], encryption [SHRS17], FHE [LTV12, BLLN13]

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However, some schemes require a full trapdoor $\mathbf{B} = \left[\begin{array}{c|c} g & -f \\ \hline G & -F \end{array}\right]$:

- Hash-then-sign [PFH$^+$17], IBE [DLP14], HIBE [CG17]
- More generally, anything based on trapdoor sampling [GPV08]

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Problem: Given $f, g \in \mathbb{Z}[x]/(x^d+1)$, find $F, G \in \mathbb{Z}[x]/(x^d+1)$ such that:

$$f \cdot G - g \cdot F = q$$

Fun fact

If we can solve the problem projected over $\mathcal{Z}_{d/2}$, i.e.:

$$\mathrm{N}_{\mathcal{Z}_d/\mathcal{Z}_{d/2}}(f) \cdot G' - \mathrm{N}_{\mathcal{Z}_d/\mathcal{Z}_{d/2}}(g) \cdot F' = 1$$

for some F', G', then we have this relationship over $\mathcal{Z}_d$:

$$f \cdot (f^{\times} G') - g \cdot (g^{\times} F') = 1$$

This leads to a simple algorithm:

1. Project
2. Solve
3. Lift

Outline of the Solver

$$\begin{array}{ccc}
\mathcal{Z}_d & \ni & f, g \\
\cup\nmid & & \\
\mathcal{Z}_{d/2} & & \\
\cup\nmid & & \\
\mathcal{Z}_{d/4} & & \\
\cup\nmid & & \\
\vdots & & \\
\cup\nmid & & \\
\mathbb{Z} & &
\end{array}$$

Outline of the Solver

$$
\begin{array}{ccc}
\mathcal{Z}_d & \ni & f, g \\
\cup\!\! \shortmid & & \downarrow \\
\mathcal{Z}_{d/2} & \ni & \mathrm{N}_{\mathcal{Z}_d/\mathcal{Z}_{d/2}}(f), \mathrm{N}_{\mathcal{Z}_d/\mathcal{Z}_{d/2}}(g) \\
\cup\!\! \shortmid & & \\
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\rotatebox{90}{$\subsetneq$} & & \downarrow \\
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\rotatebox{90}{$\supsetneq$} & & \downarrow \\
\mathcal{Z}_{d/4} & \ni & \mathrm{N}_{\mathcal{Z}_d/\mathcal{Z}_{d/4}}(f), \mathrm{N}_{\mathcal{Z}_d/\mathcal{Z}_{d/4}}(g) \\
\rotatebox{90}{$\supsetneq$} & & \downarrow \\
\vdots & \vdots & \vdots \\
\rotatebox{90}{$\supsetneq$} & & \downarrow \\
\mathbb{Z} & \ni & \mathrm{N}_{\mathcal{Z}_d/\mathbb{Z}}(f), \mathrm{N}_{\mathcal{Z}_d/\mathbb{Z}}(g)
\end{array}
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\vdots & \vdots & \vdots & & \\
\cup\!\nmid & & \downarrow & & \\
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\rotatebox{90}{$\subsetneq$} & & \downarrow & & \uparrow \\
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$$

At each lower level:

- The coefficients grow (in bitsize) by a factor 2...
- ... but the number of coefficients is divided by 2.

Space-saving trick: recompute lazily $\mathrm{N}^i(f), \mathrm{N}^i(g)$ at each step

- Allows a linear time-memory trade-off by a factor $\ell = \log n$

Toy Example

```
sage: f8, g8
-x^7 + 3*x^6 - x^4 + 4*x^3 + 6*x^2 - 2*x - 4,
x^7 - x^6 - 2*x^5 - 4*x^3 - 3*x^2 - x + 7
sage: f4, g4
-51*x^3 + 51*x^2 - 54*x - 17, -33*x^3 - 4*x^2 - 47*x + 57
sage: f2, g2
-2049*x + 3196, -1576*x + 6335
sage: f1, g1
14412817, 42616001
sage: F1, G1
5126443, 15157932
sage: F2, G2
2495*x - 399, 3844*x - 2025
sage: F4, G4
-22*x^3 + 39*x^2 - 23*x - 14, -x^3 - 45*x + 5
sage: F8, G8
-x^7 - x^5 + 3*x^4 + 3*x^3 - 3*x^2 + 4,
2*x^7 - x^6 - x^5 - x^4 - 3*x^3 + x^2 + x - 4
```

Performances

Method	Time complexity[1]	Space complexity[1]
Resultant [HHGP+03]	$\tilde{O}(d(d^2 + B))$	$O(d^2 B)$
HNF [SS11]	$\tilde{O}(d^3 B)$	$O(d^2 B)$
This work (Fast)	$O((dB)^{\log_2 3} \log d)$ [Kara] $\tilde{O}(dB)$ [SchöStr]	$O(d(B + \log d) \log d)$
This work (Compact)	$O((dB)^{\log_2 3} \log^2 d)$ [Kara] $\tilde{O}(dB)$ [SchöStr]	$O(d(B + \log d))$

We gain in practice:

- a factor 100 in memory (3 MB $\rightarrow$ 30 kB)
- a factor 100 in time (2 sec. $\rightarrow$ 20 msec.)

[1] $B = \log_2 ||(f, g)||$

Problem: Given $\mathbf{A} \in \mathcal{R}^{n \times n}$, compute $\mathbf{B}_1, \ldots, \mathbf{B}_k \in \mathcal{R}^{n \times n}$ such that

$$\mathbf{A}\mathbf{A}^{\star} + \sum_i \mathbf{B}\mathbf{B}^{\star} = C \cdot \mathbf{I}_n$$

Algorithmic solutions:

- $\mathcal{R} = \mathbb{R}$, $k = 1$: Cholesky [Pei10]
- $\mathcal{R} = \mathbb{R}[x]/(\varphi)$, $k = 1$: Babylonian method [DN12]
- $\mathcal{R} = \mathbb{Z}$, $k = O(1)$: `ia.cr/2019/320`
- $\mathcal{R} = \mathbb{Z}[x]/(x^d + 1)$, $k = O(\log d)$: This talk + `ia.cr/2019/320`

Simplified Problem

Simplified problem: Given $a \in \mathbb{Z}[x]/(x^d+1)$, compute polynomials $b_1, \ldots, b_{\log_2(d)} \in \mathbb{Z}[x]/(x^d+1)$ such that for some constant C:

$$a\overline{a} + \sum_i b_i \overline{b_i} = C,$$

where $\overline{\cdot}$ denotes the Hermitian adjoint (in our case, $\overline{a}(x) = a(x^{-1})$).

Attempt 1: Galois conjugation and Hermitian adjoint compose nicely:

$$\mathrm{Tr}_{\mathcal{Q}_d/\mathcal{Q}_{d/2}}(a\overline{a}) = a\overline{a} + (a\overline{a})^\times = a\overline{a} + a^\times \overline{a^\times} \in \mathcal{Z}_{d/2}$$

- ☺ We have projected the problem over $\mathcal{Z}_{d/2}$.
- ☹ Unfortunately repeating this trick doesn't scale well.

A Scalable Solution

Attempt 2: Let $a\overline{a} = g$; g is self-adjoint so we can write $g = g_{\text{low}} + \overline{g_{\text{low}}}$. Let $b(x) = 1 - x \cdot g_{o,\text{low}}(x^2)$, then:

$$\begin{aligned} g + b\overline{b} &= g_e(x^2) + x \cdot g_{o,\text{low}}(x^2) + \overline{x \cdot g_{o,\text{low}}(x^2)} \\ &\quad + (1 - x \cdot g_{o,\text{low}}(x^2)) \cdot \overline{(1 - x \cdot g_{o,\text{low}}(x^2))} \\ &= (1 + g_e + g_{o,\text{low}} \cdot \overline{g_{o,\text{low}}})(x^2) \end{aligned}$$

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- ☺ This trick scales well with repetition.
- ☹ It incurs a growth on the coefficients' sizes...
- ☺ ... but composes nicely with gadget decomposition:
  - ➻ We write $g = g_0 + 2 \cdot g_1 + \cdots + 2^k g_k$,
  - ➻ Then we apply this trick on each g_i.

  This effectively mitigates the size growth.

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  This effectively mitigates the size growth.

Consequence: We can compute $b_1, \ldots, b_k$ in $\mathcal{Z}_d$ such that

$$a\bar{a} + \sum_i b_i \overline{b_i} = C,$$

with $k = \tilde{O}(\log \|g\|_\infty + \log d)$.

Problem 3 - Efficient Lattice Decoding

Problem: Given $\mathbf{B} \in \mathcal{Z}_d^{n \times n}$ and $\mathbf{c} \in \mathrm{Span}_{\mathcal{Q}_d}(\mathbf{B})$, compute $\mathbf{v} \in \Lambda(\mathbf{B})$ such that

$$||\mathbf{v} - \mathbf{c}|| \text{ is small.}$$

Equivalent: Given $\mathbf{B} \in \mathcal{Z}_d^{n \times n}$ and $\mathbf{t} \in \mathcal{Q}_d^n$, compute $\mathbf{z} \in \mathcal{Z}_d^n$ such that

$$||(\mathbf{z} - \mathbf{t}) \cdot \mathbf{B}|| \text{ is small.}$$

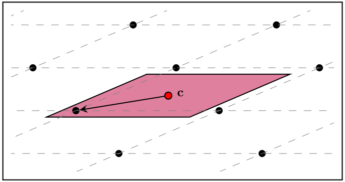
Algorithmic solutions:

- High quality, $O((nd)^2)$ operations — (Randomized) nearest plane [Bab85, GPV08]
- Lower quality, $O(n^2 d \log d)$ operations — (Randomized) round-off [Bab85, Pei10]
- High quality, $O(n^2 d \log d)$ operations — Fast Fourier orthogonalization `ia.cr/2015/1014`

Round-Off Algorithm:

1. $\mathbf{t} \leftarrow \mathbf{c} \cdot \mathbf{B}^{-1}$
2. $\mathbf{z} \leftarrow \lfloor \mathbf{t} \rceil$
3. Output $\mathbf{v} \leftarrow \mathbf{z} \cdot \mathbf{B}$

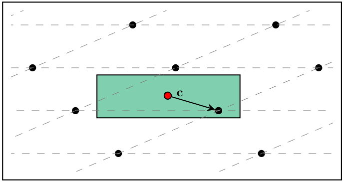
Output:



Nearest Plane Algorithm:[1]

1. $\mathbf{t} \leftarrow \mathbf{c} \cdot \mathbf{B}^{-1}$
2. For $j = n$ down to 1:
   1. $\hat{t}_j \leftarrow t_j + \sum_{i>j} (t_i - z_i) \cdot L_{i,j}$
   2. $z_j \leftarrow \lfloor \hat{t}_j \rceil$
3. Output $\mathbf{v} \leftarrow \mathbf{z} \cdot \mathbf{B}$

Output:

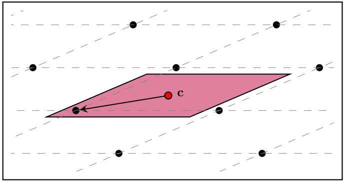


[1] Requires precomputing the Gram-Schmidt orthogonalisation (GSO) of $\mathbf{B}$: $\mathbf{B} = \mathbf{L} \cdot \tilde{\mathbf{B}}$..

Round-Off Algorithm:

1. $\mathbf{t} \leftarrow \mathbf{c} \cdot \mathbf{B}^{-1}$
2. $\mathbf{z} \leftarrow \lfloor \mathbf{t} \rceil$
3. Output $\mathbf{v} \leftarrow \mathbf{z} \cdot \mathbf{B}$

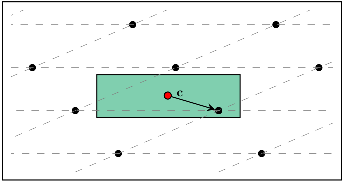
Output:



Nearest Plane Algorithm:[1]

1. $\mathbf{t} \leftarrow \mathbf{c} \cdot \mathbf{B}^{-1}$
2. For $j = n$ down to 1:
   1. $\hat{t}_j \leftarrow t_j + \sum_{i>j}(t_i - z_i) \cdot L_{i,j}$
   2. $z_j \leftarrow \lfloor \hat{t}_j \rceil$
3. Output $\mathbf{v} \leftarrow \mathbf{z} \cdot \mathbf{B}$

Output:



[1] Requires precomputing the Gram-Schmidt orthogonalisation (GSO) of **B**: $\mathbf{B} = \mathbf{L} \cdot \tilde{\mathbf{B}}$.

Tricks and Tips (1/2)

Consider the simplified case where we want this to be small:

$$(z - t) \cdot b$$

Using the ring isomorphism $\mathcal{Q}_d \cong (\mathcal{Q}_{d/2})^2$, this is equivalent to:

$$\left[\begin{array}{c|c} z_e - t_e & z_o - t_o \end{array}\right] \cdot \underbrace{\left[\begin{array}{c|c} b_e & b_o \\ \hline x b_o & b_e \end{array}\right]}_{\mathbf{B}}$$

Why this is nice:

- We can orthogonalize the second row of **B** w.r.t. to the first one:

$$\tilde{\mathbf{b}}_2 \leftarrow \mathbf{b}_2 - \underbrace{\frac{\langle \mathbf{b}_2, \mathbf{b}_1 \rangle}{\mathbf{b}_2, \mathbf{b}_1}}_{L_{2,1}} \cdot \mathbf{b}_1$$

- We can apply this "break and orthogonalize" trick recursively.
- This structured decomposition then allows a faster nearest plane algorithm.

Tricks and Tips (2/2)

Additional tricks:

- **Equivalent decomposition:**

$$\underbrace{(\mathbf{B} = \mathbf{L} \cdot \tilde{\mathbf{B}})}_{\text{GSO}} \Longleftrightarrow \underbrace{(\mathbf{B} \cdot \mathbf{B}^\star = \mathbf{L} \cdot \tilde{\mathbf{B}}\tilde{\mathbf{B}}^\star \cdot \mathbf{L}^\star)}_{\text{LDL decomposition}}$$

  The LDL decomposition is more amenable to a recursive application of our trick; this yields a complexity $O(d \log^2 d)$.

- **Working only in the FFT domain:** Discarding useless conversions further reduces the total complexity to $O(d \log d)$.

Summary (non-exhaustive)

Speed-ups in the presence of a ring:

- Most of efficient lattice-based cryptography

Speed-ups in the presence of tower of rings (**this talk**):

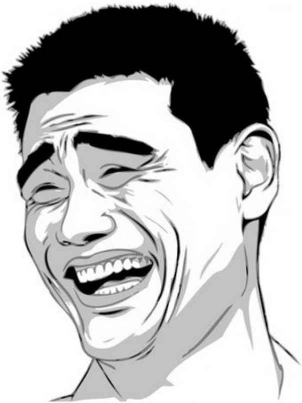
- Using ring isomorphisms: `ia.cr/2015/1014`
- Using the field norm: `ia.cr/2019/015`
- Using trace-like properties: `ia.cr/2019/230`

Exploiting automorphisms:

- Homomorphic encryption
- Zero-Knowledge proofs [dPLS18]

Key takeaway

If you cannot trivially exploit the presence of a ring...



... use its particular structure!

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