Lattice-Based Signatures

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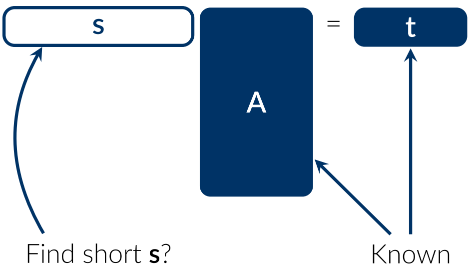
PQCifris 2022 Workshop

https://tprest.github.io/pdf/slides/pqcifris-2022.pdf

Introduction

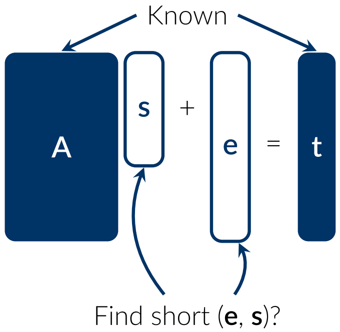
Lattice assumptions

Short integer solution (SIS)



- Used in dense/surjective regime
- Gets easier when $\|\mathbf{s}\|$ increases
- Also hard to solve approximately [CGM19]

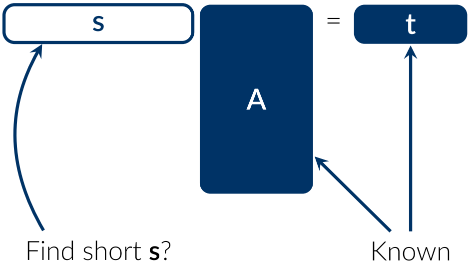
Learning with errors (LWE)



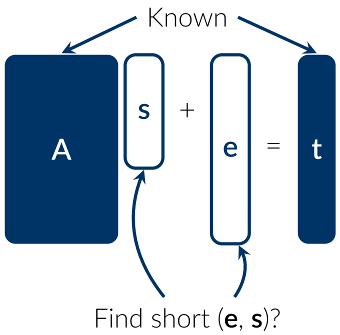
- Used in sparse/injective regime
- Gets harder when $\|(\mathbf{s}, \mathbf{e})\|$ increases
- Also hard to solve approximately

Lattice assumptions

Short integer solution (SIS)



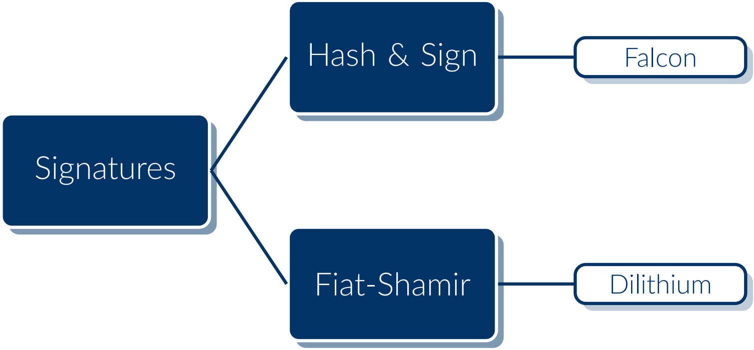
Learning with errors (LWE)



NTRU

Given $h \in \mathcal{R}_q = \mathbb{Z}_q[x]/(\varphi)$, find small f, g such that $g \cdot f^{-1} = h$ (i.e. $\begin{bmatrix} f & g \end{bmatrix} \cdot \begin{bmatrix} h \\ -1 \end{bmatrix} = \mathbf{0}$)

Roadmap



Part I: Hash & Sign

- High-level principle
- Choice of lattice class
- Choice of sampler

Part II: Fiat-Shamir

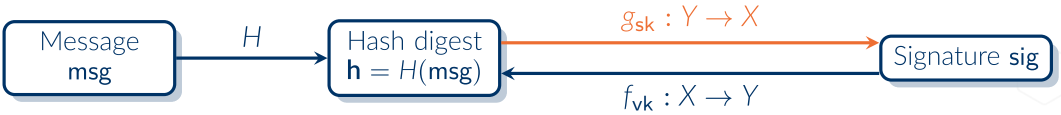
- High-level principle
- Choice of lattice class
- Ninja tricks
- Choice of distribution

Warning:

- Some mathematics are oversimplified
- Does not cover recent schemes based on the lattice isomorphism problem (LIP)

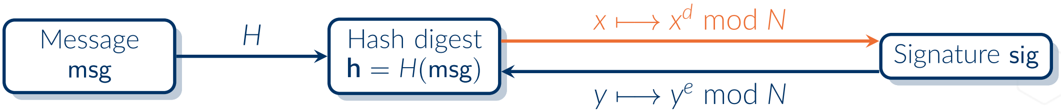
Hash-then-Sign

Signatures based on *Hash-then-Sign*



- **The signer** computes $\mathbf{h} = H(\mathsf{msg})$, then $\mathsf{sig} = g_{\mathsf{sk}}(\mathbf{h})$ using the signing key **sk**.
- **The verifier** computes $\mathbf{h} = H(\mathsf{msg})$, then $\mathbf{h}' = f_{\mathsf{vk}}(\mathsf{sig})$ using the verification key **vk**, and checks that the results match (i.e. $\mathbf{h}' = \mathbf{h}$).

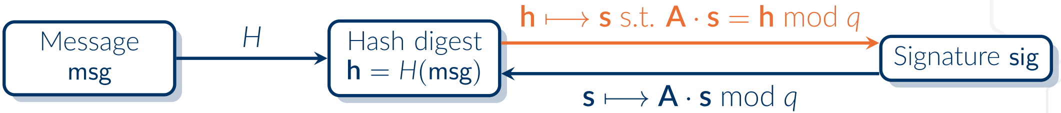
The case of RSA signatures



Example with RSA signatures:

- $g_{\mathbf{sk}}(x) = x^d \bmod N$, and $f_{\mathbf{vk}}(y) = y^e \bmod N$.
- $e \cdot d = 1 \bmod \phi(N)$

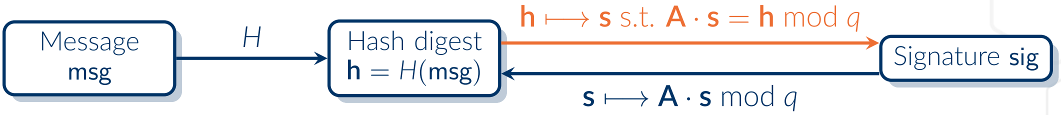
The case of lattices (first attempt)



First attempt with lattice (not secure):

- *Verification key:* **vk** is a (pseudo)random matrix $\mathbf{A} \in \mathcal{R}_q^{n \times m}$.
- *Signing key:* **sk** is a short matrix $\mathbf{B} \in \mathcal{R}_q^{m \times m}$ such that $\mathbf{A} \cdot \mathbf{B} = \mathbf{0} \bmod q$.
- *Signing:*
  1. Hash **msg** to a point $\mathbf{h} \in \mathcal{R}_q^n$.
  2. Compute $\mathbf{c} \in \mathcal{R}_q^m$ s.t. $\mathbf{A} \cdot \mathbf{c} = \mathbf{h}$.
  3. Compute $\mathbf{v} \in \mathbf{B} \cdot \mathcal{R}_q^m$ close to $\mathbf{v}$ (the hard part, see next slide)
  4. The signature is $\mathbf{s} := \mathbf{c} - \mathbf{v}$
- *Verification:*
  1. Check that $\mathbf{A} \cdot \mathbf{s} = \mathbf{h}$.
  2. Check that $\mathbf{s}$ is short (say, $\|\mathbf{s}\|_2$ is small).

The case of lattices (first attempt)



First attempt with lattice (not secure):

- *Verification key:* vk is a (pseudo)random matrix $\mathbf{A} \in \mathcal{R}_q^{n \times m}$.
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- *Verification:*
  1. Check that $\mathbf{A} \cdot \mathbf{s} = \mathbf{h}$.
  2. Check that $\mathbf{s}$ is short (say, $\|\mathbf{s}\|_2$ is small).

Big questions:

- How do we generate a suitable keypair $(\mathbf{A}, \mathbf{B})$?
- How do we compute $\mathbf{v}$ close to $\mathbf{v}$?

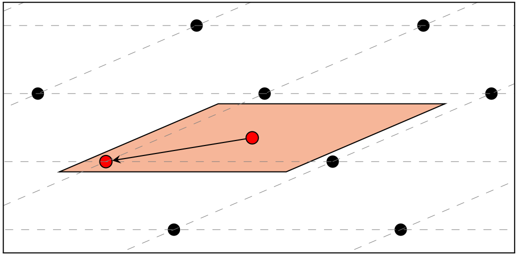
Computing a lattice point v close to the target c

For NearestPlane, the Gram-Schmidt orthogonalization $\mathbf{B} = \mathbf{L} \cdot \tilde{\mathbf{B}}$ is precomputed.

RoundOff(B, c)

1. $\mathbf{t} \leftarrow \mathbf{c} \cdot \mathbf{B}^{-1}$
2. For $j \in \{n, \ldots, 1\}$:
   1. $z_j \leftarrow \lceil t_j \rfloor$
3. Return $\mathbf{v} := \mathbf{z} \cdot \mathbf{B}$

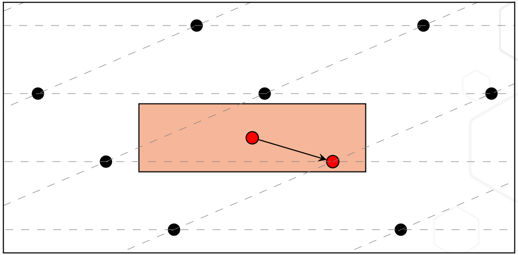
$\Downarrow$



NearestPlane(B, L, c)

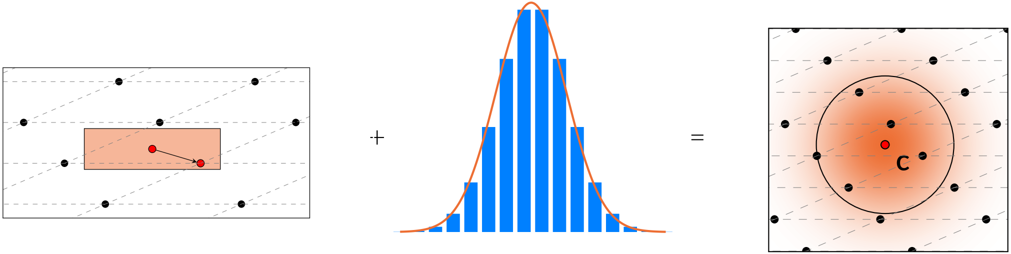
1. $\mathbf{t} \leftarrow \mathbf{c} \cdot \mathbf{B}^{-1}$
2. For $j \in \{n, \ldots, 1\}$:
   1. $z_j \leftarrow \lceil t_j + \sum_{i>j} (t_1 - z_i) L_{i,j} \rfloor$
3. Return $\mathbf{v} := \mathbf{z} \cdot \mathbf{B}$

$\Downarrow$



Trapdoor sampling [GPV08]

Problem: the distribution of signatures may leak the shape of **B**
Solution: randomize the solving procedure with Gaussians



Computing a suitable (A, B) – NTRU trapdoors

NTRU trapdoors

Let $f, g, F, G \in \mathcal{R}$ such that:

$$fG - gF = q \tag{1}$$

$$h := g/f \bmod q \tag{2}$$

We set $\mathbf{A} = \begin{bmatrix} 1 & h \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} g & G \\ -f & -F \end{bmatrix}$.

Computing a suitable (A, B) – NTRU trapdoors

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$$fG - gF = q \tag{1}$$

$$h := g/f \bmod q \tag{2}$$

We set $\mathbf{A} = \begin{bmatrix} 1 & h \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} g & G \\ -f & -F \end{bmatrix}$.

- **Pseudorandomness of A:** NTRU assumption.
- **Orthogonality:** One can easily show that $\mathbf{A} \cdot \mathbf{B} = \mathbf{0} \bmod q$.
- **Shortness of B:** Given (f, g), one can compute suitable (F, G) such that

$$\|(F, G)\| \approx \underbrace{\frac{q}{\|(f, g)\|}}_{\text{component} \perp (f,g)} + \underbrace{\sqrt{\frac{d}{12}} \cdot \|(f, g)\|}_{\text{component} \parallel (f,g)} \tag{3}$$

Computing a suitable (A, B) – [MP12] trapdoors

Gadget matrices

We define $\mathbf{g}, \mathbf{B}$ such that $\mathbf{g} \cdot \mathbf{B} = \mathbf{0} \bmod q$:

→ $$\mathbf{g} = (1, b, b^2, \ldots, b^{k-1}) \text{ and } \mathbf{B} = \begin{bmatrix} b & & & q_0 \\ -1 & \ddots & & \vdots \\ & \ddots & b & q_{k-2} \\ & & -1 & q_{k-1} \end{bmatrix}, \text{ where } q = \sum_i q_i b^i$$

→ The "gadget matrix" $\mathbf{G} = \mathbf{I}_n \otimes \mathbf{g}$ and $\mathbf{G}^\perp = \mathbf{I}_n \otimes \mathbf{B}$ also satisfy $\mathbf{G} \cdot \mathbf{G}^\perp = \mathbf{0} \bmod q$.

Generating a Micciancio-Peikert trapdoor

→ Set $\bar{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{A}} & \mathbf{I} \end{bmatrix}$, where $\hat{\mathbf{A}}$ is a uniformly random matrix.

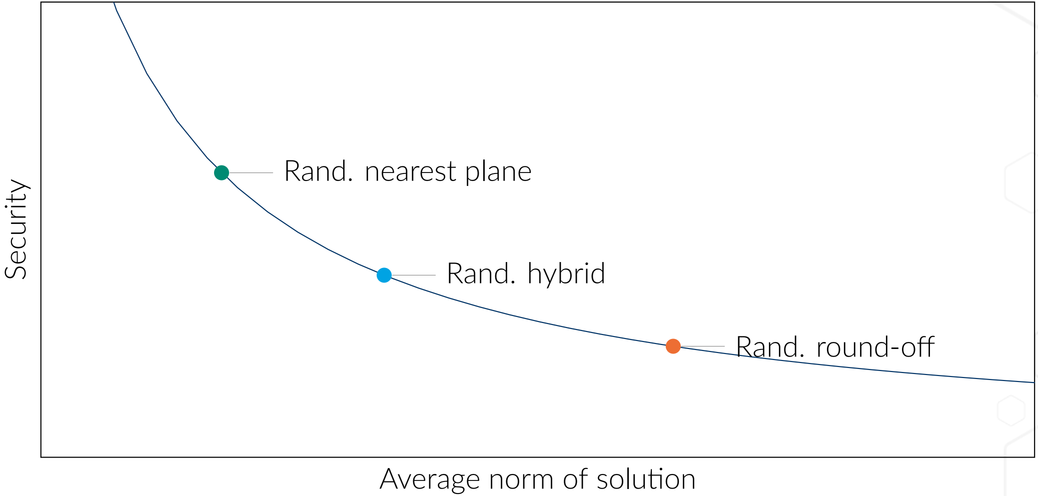
→ Generate a short random matrix $\mathbf{R}$

→ Set $\mathbf{A} = [\bar{\mathbf{A}} \quad \mathbf{G} - \bar{\mathbf{A}} \cdot \mathbf{R}]$ and $\mathbf{B} = \begin{bmatrix} \mathbf{R} \\ \mathbf{I} \end{bmatrix} \cdot \mathbf{G}^\perp$.

Pseudorandomness (under LWE), orthonogality and shortness: Exercise.

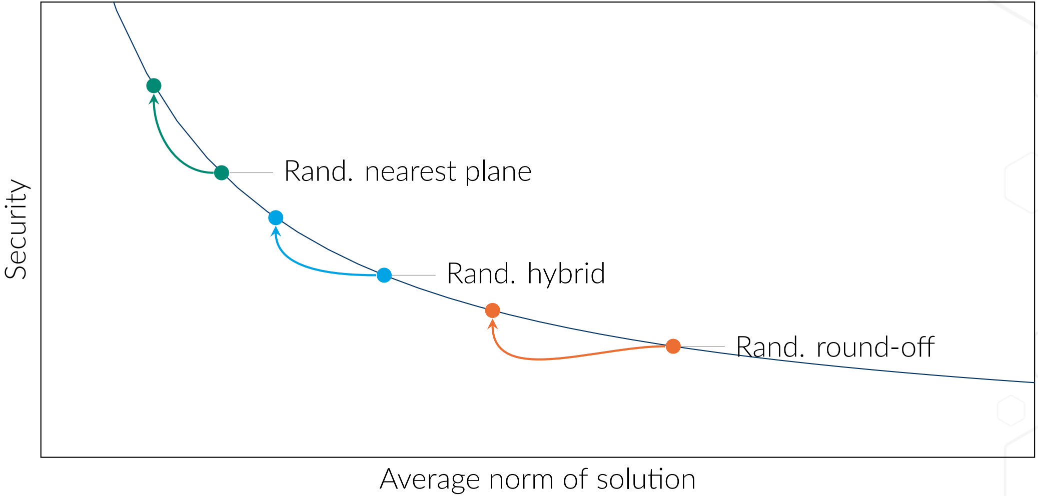
Example with NTRU trapdoors

Remember SIS (solving $\mathbf{A} \cdot \mathbf{s} = \mathbf{t}$) gets harder when $||\mathbf{s}||$ is shorter.



Example with NTRU trapdoors

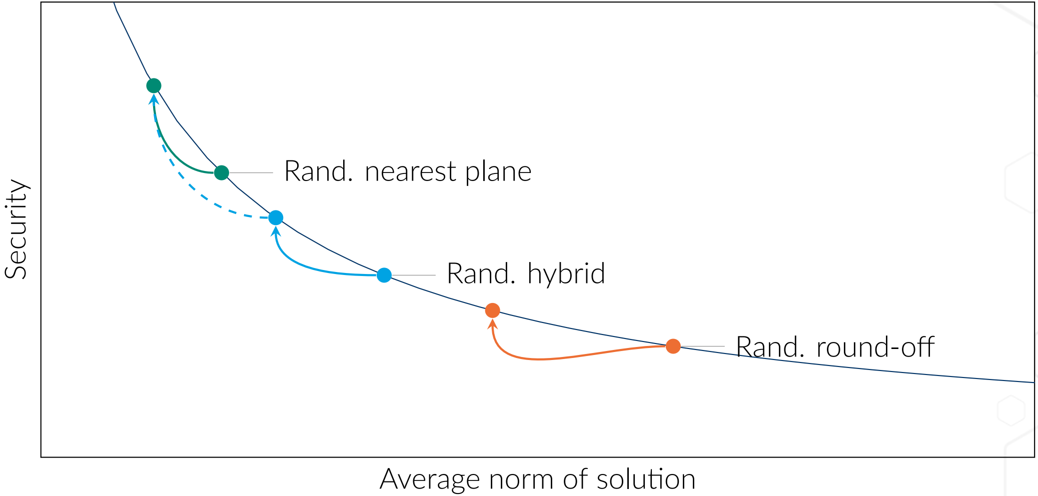
Remember SIS (solving $\mathbf{A} \cdot \mathbf{s} = \mathbf{t}$) gets harder when $\|\mathbf{s}\|$ is shorter.



→ 2017: Improved statistical analyses w/ R'enyi divergence (solid arrows)

Example with NTRU trapdoors

Remember SIS (solving $\mathbf{A} \cdot \mathbf{s} = \mathbf{t}$) gets harder when $\|\mathbf{s}\|$ is shorter.



- 2017: Improved statistical analyses w/ R'enyi divergence (solid arrows)
- 2022: Improved generation of NTRU trapdoors (dashed arrow)

Section conclusion and bibliography

Foundations

- Trapdoor sampling [GPV08]
- Micciancio-Peikert sampling [MP12]

Trapdoor samplers

- Randomised nearest plane [GPV08]
- Randomised round-off [Pei10]
- Hybrid [Pre15]
- Fast Fourier sampling [DP16]

Trapdoor lattices

- NTRU lattices [HHP+03, DLP14]
- Micciancio-Peikert trapdoors [MP12, CGM19]
- Improved NTRU trapdoors [ea22]

Efficient instantiations

- Falcon (NTRU) [PFH+17]
- Mitaka (NTRU) [EFG+22]
- (Micciancio-Peikert) [CGM19]

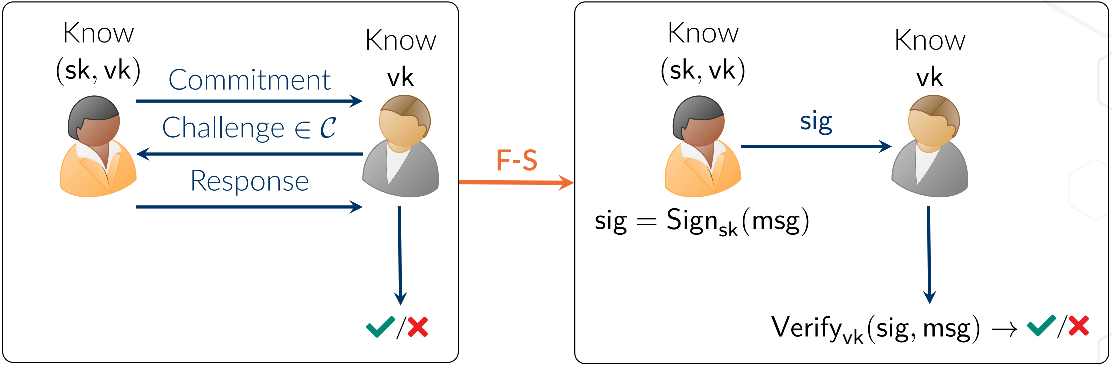
Proof techniques

- Security model [GPV08, CGM19]
- Statistical relaxations [Pre17]

Fiat-Shamir Signatures

Fiat–Shamir Signatures

(3-Move) Identification Protocol — **Signature Scheme**

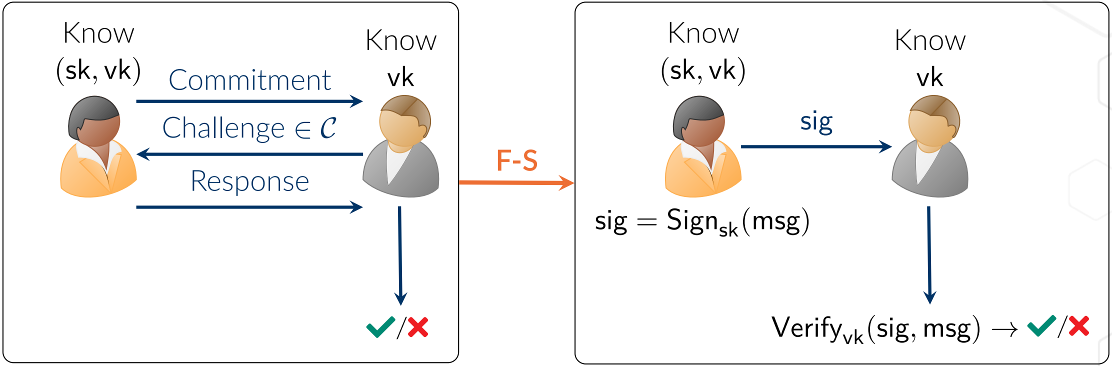


F-S refers to the Fiat-Shamir transform:

- → The challenge is now defined as $H(\text{Commitment}\|\textbf{msg})$.
- → The signature is (Commitment,Response).

Fiat–Shamir Signatures

(3-Move) Identification Protocol | **Signature Scheme**



We obtain an existentially unforgeable signature scheme in the ROM if the ID protocol is:

1. **Correct:** An honest prover can convince a verifier he knows sk
2. **Honest verifier zero-knowledge:** A valid transcript can be simulated without sk
3. **Soundness:** A dishonest prover cannot convince a verifier he knows sk

Schnorr signatures (Fiat-Shamir w/ discrete log)

Keygen($g \in \mathbb{G}$)

1. $x \leftarrow \mathbb{Z}_q^\times$ $\quad (q = |\mathbb{G}|)$
2. $h \leftarrow g^x$
3. $\mathsf{sk} := x, \mathsf{vk} := h$

Sign(msg, sk)

1. $r \leftarrow \mathbb{Z}_q^\times$
2. $u \leftarrow g^r$ (Commitment)
3. $c \leftarrow H(u \| \mathsf{msg})$ (Challenge)
4. $z \leftarrow r - cx$ (Response)
5. $\mathsf{sig} := (u, z)$

Verify(msg, vk)

1. Accept if and only if $(g^z \cdot h^c = u)$

It is easy to show:

- ✓ **Correctness**
- ✓ **HVZK**
- ✓ **Special soundness**

Note that **DSA** and **ECDSA** are very similar to this scheme.

Fiat-Shamir with SIS

Keygen($\mathbf{A} \in \mathcal{R}_q^{k \times \ell}$)

1. $\mathbf{s} \leftarrow \chi_1$ (short)
2. $\mathbf{t} \leftarrow \mathbf{A}\mathbf{s}$
3. $\mathsf{sk} := \mathbf{s}, \mathsf{vk} := \mathbf{t}$

Sign(msg, sk)

1. $\mathbf{r} \leftarrow \chi_2$ (short)
2. $\mathbf{u} \leftarrow \mathbf{A}\mathbf{r}$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathsf{msg})$
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}$
5. $\mathsf{sig} := (\mathbf{u}, \mathbf{z})$

Verify(msg, vk, sig)

1. Accept iff ($\mathbf{z}$ is short) and ($\mathbf{A}\mathbf{z} - \mathbf{c}\mathbf{t} = \mathbf{u}$).

Fiat-Shamir with SIS

Keygen($\mathbf{A} \in \mathcal{R}_q^{k \times \ell}$)

1. $\mathbf{s} \leftarrow \chi_1$ (short)
2. $\mathbf{t} \leftarrow \mathbf{A}\mathbf{s}$
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4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}$
5. $\mathsf{sig} := (\mathbf{u}, \mathbf{z})$

Verify(msg, vk, sig)

1. Accept iff ($\mathbf{z}$ is short) and ($\mathbf{A}\mathbf{z} - \mathbf{c}\mathbf{t} = \mathbf{u}$).

- ✓ Correctness
- ✗ HVZK
- ✗ Special soundness

Fiat-Shamir with SIS

Keygen$(\mathbf{A} \in \mathcal{R}_q^{k \times \ell})$

1. $\mathbf{s} \leftarrow \chi_1$ (short)
2. $\mathbf{t} \leftarrow \mathbf{As}$
3. $\mathsf{sk} := \mathbf{s}, \mathsf{vk} := \mathbf{t}$

Sign$(\mathsf{msg}, \mathsf{sk})$

1. $\mathbf{r} \leftarrow \chi_2$ (short)
2. $\mathbf{u} \leftarrow \mathbf{Ar}$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathsf{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{cs}$
5. $\mathsf{sig} := (\mathbf{u}, \mathbf{z})$

Verify$(\mathsf{msg}, \mathsf{vk}, \mathsf{sig})$

1. Accept iff ($\mathbf{z}$ is short) and ($\mathbf{Az} - \mathbf{ct} = \mathbf{u}$).

Soundness: Using rewinding:

- Transcript 1: $(\mathbf{u}, \mathbf{c}, \mathbf{z} \mid \mathbf{Az} - \mathbf{ct} = \mathbf{u})$
- Transcript 2: $(\mathbf{u}, \mathbf{c}', \mathbf{z}' \mid \mathbf{Az}' - \mathbf{c}'\mathbf{t} = \mathbf{u})$

$$[\,\mathbf{A} \,\|\, \mathbf{t}\,] \cdot \begin{bmatrix} \mathbf{z} - \mathbf{z}' \\ \mathbf{c} - \mathbf{c}' \end{bmatrix} = \mathbf{0} \qquad (4)$$

- ✓ **Correctness**
- ✗ **HVZK**
- ✓ **Special soundness** (imperfect) is satisfied, as long as $\mathbf{c}$ is short.

Fiat-Shamir with SIS

Keygen($\mathbf{A} \in \mathcal{R}_q^{k \times \ell}$)

1. $\mathbf{s} \leftarrow \chi_1$ (short)
2. $\mathbf{t} \leftarrow \mathbf{A}\mathbf{s}$
3. $\mathrm{sk} := \mathbf{s}, \mathrm{vk} := \mathbf{t}$

Sign(msg, sk)

1. $\mathbf{r} \leftarrow \chi_2$ (short)
2. $\mathbf{u} \leftarrow \mathbf{A}\mathbf{r}$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathrm{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}$
5. Rejection sampling step
6. $\mathrm{sig} := (\mathbf{u}, \mathbf{z})$

Verify(msg, vk, sig)

1. Accept iff ($\mathbf{z}$ is short) and ($\mathbf{A}\mathbf{z} - \mathbf{c}\mathbf{t} = \mathbf{u}$).

- ✓ **Correctness**
- ✓ **HVZK** requires rejection sampling.
- ✓ **Special soundness** (imperfect) is satisfied, as long as $\mathbf{c}$ is short.

Without rejection sampling, statistical attacks may recover the signing key.

Fiat-Shamir with SIS

Keygen($\mathbf{A} \in \mathcal{R}_q^{k \times \ell}$)

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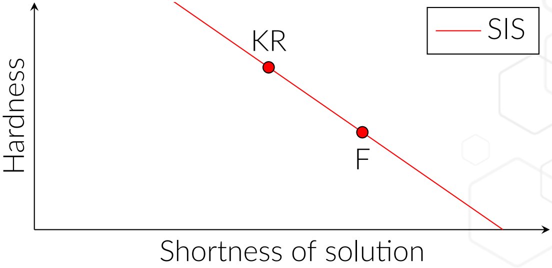
5. Rejection sampling step
6. $\mathsf{sig} := (\mathbf{u}, \mathbf{z})$

Verify(msg, vk, sig)

1. Accept iff ($\mathbf{z}$ is short) and ($\mathbf{A}\mathbf{z} - \mathbf{c}\mathbf{t} = \mathbf{u}$).

Concrete hardness:

- *Key-recovery:* SIS with a short $\mathbf{s}$
- *Forgery:* SIS with a short-ish $\mathbf{z}$



Fiat-Shamir w/ (LWE+SIS) [Lyu12]

Keygen($\mathbf{A} \in \mathcal{R}_q^{k \times \ell}$)

1. $\mathbf{s}_1, \mathbf{s}_2 \leftarrow \chi_1 \times \chi_2$ (short)
2. $\mathbf{t} \leftarrow \mathbf{A}\mathbf{s}_1 + \mathbf{s}_2$
3. $\mathsf{sk} := (\mathbf{s}_1, \mathbf{s}_2), \mathsf{vk} := \mathbf{t}$

Sign(msg, sk)

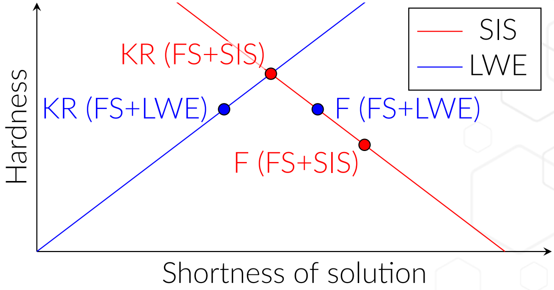
1. $\mathbf{r}_1, \mathbf{r}_2 \leftarrow \chi_3 \times \chi_4$ (short)
2. $\mathbf{u} \leftarrow \mathbf{A}\mathbf{r}_1 + \mathbf{r}_2$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathsf{msg})$ (short)
4. $\mathbf{z}_1 \leftarrow \mathbf{r}_1 - \mathbf{c}\mathbf{s}_1$
5. $\mathbf{z}_2 \leftarrow \mathbf{r}_2 - \mathbf{c}\mathbf{s}_2$
6. Rejection sampling step
7. $\mathsf{sig} := (\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2)$

Verify(msg, vk, sig)

1. Accept iff $(\mathbf{z}_1, \mathbf{z}_2)$ is short and $\mathbf{A}\mathbf{z}_1 + \mathbf{z}_2 - \mathbf{t}\mathbf{c} = \mathbf{u}$

Concrete hardness:

- *Key-recovery:* LWE with a short $\mathbf{s}$
- *Forgery:* SIS with a short-ish $\mathbf{z}$



Fiat-Shamir w/ (LWE+SIS) [Lyu12]

Keygen($\mathbf{A} \in \mathcal{R}_q^{k\times\ell}$)

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3. $\mathsf{sk} := (\mathbf{s}_1, \mathbf{s}_2), \mathsf{vk} := \mathbf{t}$

Sign(msg, sk)

1. $\mathbf{r}_1, \mathbf{r}_2 \leftarrow \chi_3 \times \chi_4$ (short)
2. $\mathbf{u} \leftarrow \mathbf{A}\mathbf{r}_1 + \mathbf{r}_2$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathsf{msg})$ (short)
4. $\mathbf{z}_1 \leftarrow \mathbf{r}_1 - \mathbf{c}\mathbf{s}_1$
5. $\mathbf{z}_2 \leftarrow \mathbf{r}_2 - \mathbf{c}\mathbf{s}_2$
6. Rejection sampling step
7. $\mathsf{sig} := (\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2)$

Verify(msg, vk, sig)

1. Accept iff $(\mathbf{z}_1, \mathbf{z}_2)$ is short and $\mathbf{A}\mathbf{z}_1 + \mathbf{z}_2 - \mathbf{t}\mathbf{c} = \mathbf{u}$

LWE also allows **two** optimisations that can be summarised by:

> "*If you are solving LWE for* $(\mathbf{A}, \mathbf{t} + \mathbf{e})$,
> *you are also solving LWE for* $(\mathbf{A}, \mathbf{t})$."

We will note MSB := "most significant bits" (the proportion may vary).

Fiat-Shamir w/ (LWE+SIS) - Optimisation 1

PQShield

Keygen($\mathbf{A} \in \mathcal{R}_q^{k\times\ell}$)

1. $\mathbf{s}_1, \mathbf{s}_2 \leftarrow \chi_1 \times \chi_2$ (short)
2. $\mathbf{t} \leftarrow \mathbf{A}\mathbf{s}_1 + \mathbf{s}_2$
3. $\mathsf{sk} := (\mathbf{s}_1, \mathbf{s}_2), \mathsf{vk} := \mathbf{t}$

Sign(msg, sk)

1. $\mathbf{r} \leftarrow \chi_3$ (short)
2. $\mathbf{u} \leftarrow \mathsf{MSB}(\mathbf{A}\mathbf{r})$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathsf{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}_1$
5. Rejection sampling step
6. $\mathsf{sig} := (\mathbf{u}, \mathbf{z})$

Verify(msg, vk, sig)

1. Accept iff $\mathbf{z}$ is short and $\mathsf{MSB}(\mathbf{Az}\text{- }\mathbf{tc}) = \mathbf{u}$

Bai-Galbraith trick [BG14]: the response sends only $\mathbf{z} := \mathbf{z}_1$ instead of $(\mathbf{z}_1, \mathbf{z}_2)$.

- → To preserve correctness, only check that $(\mathbf{Az} - \mathbf{tc})$ and $\mathbf{u}$ match on their MSBs.
- → If moderate, bit dropping only mildly affect the hardness of LWE.

Fiat-Shamir w/ (LWE+SIS) - Optimisation 2

PQSHIELD

Keygen($\mathbf{A} \in \mathcal{R}_q^{k \times \ell}$)

1. $\mathbf{s}_1, \mathbf{s}_2 \leftarrow \chi_1 \times \chi_2$ (short)
2. $\mathbf{t} \leftarrow \mathrm{MSB}(\mathbf{A}\mathbf{s}_1 + \mathbf{s}_2)$
3. $\mathrm{sk} := (\mathbf{s}_1, \mathbf{s}_2), \mathrm{vk} := \mathbf{t}$

Sign(msg, sk)

1. $\mathbf{r} \leftarrow \chi_3$ (short)
2. $\mathbf{u} \leftarrow \mathrm{MSB}(\mathbf{A}\mathbf{r})$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathrm{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}_1$
5. Rejection sampling step
6. $\mathrm{sig} := (\mathbf{u}, \mathbf{z})$

Verify(msg, vk, sig)

1. Accept iff $\mathbf{z}$ is short and $\mathrm{MSB}(\mathbf{A}\mathbf{z} - \mathbf{t}\mathbf{c}) = \mathbf{u}$

Dilithium trick [LDK+17] (naive version):
the signer drops the least significant bits of $\mathbf{t}$ during **Keygen**.

- → vk gets shorter.
- → Intuitively, this adds an error term $\mathbf{e}$ to $\mathbf{t}$
- → $\mathbf{A}\mathbf{z} - \mathbf{c}\mathbf{t} = \mathbf{u} - \underline{\mathbf{c}(\mathbf{s}_2 + \mathbf{e})}$

With mild bit dropping, the signature is valid with good probability (if it isn't, restart).

Dilithium uses a more sophisticated version of this trick.

A closer look at rejection sampling (here with SIS)

Sign(msg, sk)

1. Sample $\mathbf{r}$ uniformly in $\{-R, R\}^n$
2. $\mathbf{u} \leftarrow \mathbf{A}\mathbf{r}$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \text{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}$
5. Rejection sampling step
6. $\text{sig} := (\mathbf{u}, \mathbf{z})$

How do we choose the distribution of $\mathbf{r}$ and perform rejection sampling? Suppose:

- $\mathbf{r}$ is sampled uniformly in $\{-R, \ldots, R\}^n$
- $\mathbf{c}\mathbf{s}_1$ is guaranteed to be in $\{-S, \ldots, S\}^n$

A closer look at rejection sampling (here with SIS)

PQSHIELD

Sign(msg, sk)

1. Sample $\mathbf{r}$ uniformly in $\{-R, R\}^n$
2. $\mathbf{u} \leftarrow \mathbf{A}\mathbf{r}$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathsf{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}$
5. Rejection sampling step
6. $\mathsf{sig} := (\mathbf{u}, \mathbf{z})$

How do we choose the distribution of $\mathbf{r}$ and perform rejection sampling? Suppose:

- → $\mathbf{r}$ is sampled uniformly in $\{-R, \dots, R\}^n$
- → $\mathbf{c}\mathbf{s}_1$ is guaranteed to be in $\{-S, \dots, S\}^n$

Does a transcript $(\mathbf{u}, \mathbf{c}, \mathbf{z})$ leak information?

✗ $\mathbf{z} \notin \{-R, \dots, R\}^n \Rightarrow \mathbf{z}$ leaks the "direction" of $\mathbf{c}\mathbf{s}_1$

✓ $\mathbf{z} \in \{-(R-S), \dots, (R-S)\}^n \Rightarrow \mathbf{z}$ leaks nothing. Indeed, for any $\mathbf{z}^*$ in this set:

$$\mathbb{P}[\mathbf{r} - \mathbf{c}\mathbf{s}_1 = \mathbf{z}^*] = \mathbb{P}[\mathbf{r} = \underbrace{\mathbf{z}^* + \mathbf{c}\mathbf{s}_1}_{\in\{-R,\dots,R\}^n}] = \frac{1}{(2R+1)^n}$$

A closer look at rejection sampling (here with SIS)

Sign(msg, sk)

1. Sample $\mathbf{r}$ uniformly in $\{-R, R\}^n$
2. $\mathbf{u} \leftarrow \mathbf{Ar}$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathsf{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{cs}$
5. Rejection sampling step
6. $\mathsf{sig} := (\mathbf{u}, \mathbf{z})$

How do we choose the distribution of $\mathbf{r}$ and perform rejection sampling? Suppose:

- → $\mathbf{r}$ is sampled uniformly in $\{-R, \ldots, R\}^n$
- → $\mathbf{cs}_1$ is guaranteed to be in $\{-S, \ldots, S\}^n$

Does a transcript $(\mathbf{u}, \mathbf{c}, \mathbf{z})$ leak information?

- ✗ $\mathbf{z} \notin \{-R, \ldots, R\}^n \Rightarrow \mathbf{z}$ leaks the “direction” of $\mathbf{cs}_1$
- ✗ $\mathbf{z} \in \{-R, \ldots, R\}^n \backslash \{-(R-S), \ldots, (R-S)\}^n \Rightarrow$ more subtle but also leaks
- ✓ $\mathbf{z} \in \{-(R-S), \ldots, (R-S)\}^n \Rightarrow \mathbf{z}$ leaks nothing. Indeed, for any $\mathbf{z}^*$ in this set:

$$\mathbb{P}[\mathbf{r} - \mathbf{cs}_1 = \mathbf{z}^*] = \mathbb{P}[\mathbf{r} = \underbrace{\mathbf{z}^* + \mathbf{cs}_1}_{\in\{-R,\ldots,R\}^n}] = \frac{1}{(2R+1)^n}$$

A closer look at rejection sampling (here with SIS)

Sign(msg, sk)

1. Sample $\mathbf{r}$ uniformly in $\{-R, R\}^n$
2. $\mathbf{u} \leftarrow \mathbf{A}\mathbf{r}$
3. $\mathbf{c} \leftarrow H(\mathbf{u} \| \mathrm{msg})$ (short)
4. $\mathbf{z} \leftarrow \mathbf{r} - \mathbf{c}\mathbf{s}$
5. If $\|\mathbf{z}\|_\infty > R - S$, goto 1
6. $\mathrm{sig} := (\mathbf{u}, \mathbf{z})$

How do we choose the distribution of $\mathbf{r}$ and perform rejection sampling? Suppose:

- → $\mathbf{r}$ is sampled uniformly in $\{-R, \ldots, R\}^n$
- → $\mathbf{c}\mathbf{s}_1$ is guaranteed to be in $\{-S, \ldots, S\}^n$

Does a transcript $(\mathbf{u}, \mathbf{c}, \mathbf{z})$ leak information?

- ✗ $\mathbf{z} \notin \{-R, \ldots, R\}^n \Rightarrow \mathbf{z}$ leaks the "direction" of $\mathbf{c}\mathbf{s}_1$
- ✗ $\mathbf{z} \in \{-R, \ldots, R\}^n \backslash \{-(R-S), \ldots, (R-S)\}^n \Rightarrow$ more subtle but also leaks
- ✓ $\mathbf{z} \in \{-(R-S), \ldots, (R-S)\}^n \Rightarrow \mathbf{z}$ leaks nothing. Indeed, for any $\mathbf{z}^*$ in this set:

$$\mathbb{P}[\mathbf{r} - \mathbf{c}\mathbf{s}_1 = \mathbf{z}^*] = \mathbb{P}[\mathbf{r} = \underbrace{\mathbf{z}^* + \mathbf{c}\mathbf{s}_1}_{\in\{-R,\ldots,R\}^n}] = \frac{1}{(2R+1)^n}$$

Accept if $\mathbf{z} \in \{-(R-S), \ldots, (R-S)\}^n$. This happens w/ prob. $\approx \left(1 - \frac{S}{R}\right)^n \leq \exp\left(-\frac{S}{nR}\right)$.

Section conclusion and bibliography

Foundations (FSwA)

- Using SIS [Lyu09]
- Using SIS + LWE [Lyu12]

Ninja tricks

- Cutting |**sig**| [BG14]
- Cutting |**vk**| [LDK+17]

Distributions

- In-depth survey [DFPS22]
- Bimodal Gaussians [DDLL13]

Efficient instantiations

- Dilithium [LDK+17]
- qTESLA [BAA+17]
- BLISS [DDLL13]

Thank You!

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