The Rényi Divergence and Security Proofs

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THALES

Introduction

What is the Rényi divergence and why use it?

Security proofs involving distributions:

- **The standard approach:** use the statistical distance Δ.
  - Take a hard problem relying on some ideal distribution $\mathcal{Q}$,
  - Replace $\mathcal{Q}$ by a "real-life" distribution $\mathcal{P}$,
  - If $\Delta(\mathcal{P}, \mathcal{Q})$ is small enough, we win: the problem is still hard.
- **Lattice-based cryptography:** often relevant to replace SD by *Rényi divergence*.
  - Sharper parameters [LSS14, LPSS14, BLL+15, BGM+16, Pre17, HLS17]
  - KEMs distributions [ADPS16, BCD+16]
  - Reduction between LWE problems [AD17]

This presentation:

1. Formalize and optimize the use of the Rényi divergence in security proofs ⇒ Section 2.
2. More applications of the Rényi divergence to lattice-based cryptography ⇒ Section 3.
3. A brief discussion on open problems ⇒ Section 4.

Based on [Pre17].

Theory

The Rényi Divergence

Definition. For $a \in (1, +\infty)$, the Rényi divergence between two distributions $\mathcal{P}, \mathcal{Q}$ is

$$R_a(\mathcal{P}\|\mathcal{Q}) = \left(\sum_{x \in Supp(\mathcal{P})} \frac{\mathcal{P}(x)^a}{\mathcal{Q}(x)^{a-1}} \right)^{\frac{1}{a-1}}$$

Motivation. We consider a cryptographic scheme doing q queries to a distribution $\mathcal{D}_i$ ($i \in \{0, 1\}$), we note ε_i the probability of an event breaking the scheme.

- With the statistical distance:

$$\varepsilon_0 \geq \varepsilon_1 - q\Delta(\mathcal{D}_1, \mathcal{D}_0) \qquad \boxed{\Delta \leq 2^{-\lambda} \Rightarrow \text{we win}}$$

- With the Rényi divergence:

$$\varepsilon_0 \geq \varepsilon_1^{\frac{a}{a-1}} / R_a(\mathcal{D}_1\|\mathcal{D}_0)^q \qquad \boxed{(a \geq \lambda)\ \&\ (\log R_a \leq 1/q) \Rightarrow \text{we win}}$$

Observation. For "equal" values ($\log R_a \approx \Delta$), Rényi divergence is more interesting when $q \ll 2^\lambda$ [BLL+15]. And typically:

- $128 \leq \lambda \leq 256$
- $1 \leq q \leq 2^{64}$

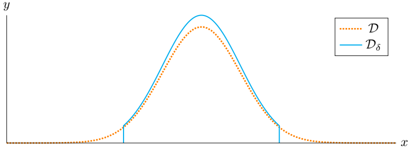
The first and second lemmas

1. **Tailcut.** Let $\delta > 0$ such that $\frac{\mathcal{D}_\delta}{\mathcal{D}} \leq 1 + \delta$. For $a \in (1, \infty]$:
   - $R_a(\mathcal{D}_\delta || \mathcal{D}) \leq (1 + \delta)^{a/a-1}$

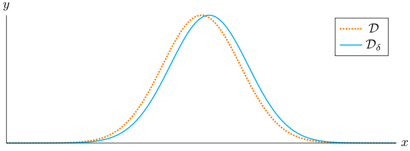
   Example: $\mathcal{D}_\delta$ is a tailcut of $\mathcal{D}$ (discard a set S such that $\mathcal{D}(S) \leq \delta$).

2. **Relative error.** Suppose $\mathrm{Supp}(\mathcal{D}_\delta) = \mathrm{Supp}(\mathcal{D})$.
   Let $\delta > 0$ such that $1 - \delta \leq \frac{\mathcal{D}_\delta}{\mathcal{D}} \leq 1 + \delta$. For $a \in (1, \infty)$:
   - $R_a(\mathcal{D}_\delta || \mathcal{D}) \leq \left(1 + \frac{a(a-1)\delta^2}{2(1-\delta)^{a+1}}\right)^{\frac{1}{a-1}} \underset{\delta \to 0}{\sim} 1 + \frac{a\delta^2}{2}$

   Example: $\mathcal{D}_\delta$ implements $\mathcal{D}$ with finite precision (relative error δ).



Tailcut lemma usecase



Relative error lemma usecase

The third lemma

The max-log distance. Introduced in [MW17].[1]
For two distributions $\mathcal{P}$ and $\mathcal{Q}$ over the same support S:

$$\Delta_{\mathrm{ML}}(\mathcal{P}, \mathcal{Q}) = \max_{x \in S} |\log \mathcal{P}(x) - \log \mathcal{Q}(x)|$$

Unlike the Rényi divergence, it is a distance, so it verifies the:

- Triangle inequality: $\Delta_{\mathrm{ML}}(\mathcal{P}, \mathcal{R}) \leq \Delta_{\mathrm{ML}}(\mathcal{P}, \mathcal{Q}) + \Delta_{\mathrm{ML}}(\mathcal{Q}, \mathcal{R})$
- Symmetry: $\Delta_{\mathrm{ML}}(\mathcal{P}, \mathcal{Q}) = \Delta_{\mathrm{ML}}(\mathcal{Q}, \mathcal{P})$

[MW17] essentially states that $\boxed{\Delta_{\mathrm{ML}} \leq 2^{-\lambda/2} \Rightarrow \text{we win}}$.

[1] Actually similar to the differential privacy.

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[MW17] essentially states that $\boxed{\Delta_{\mathrm{ML}} \leq 2^{-\lambda/2} \Rightarrow \text{we win}}$.

3. **A reverse Pinsker inequality.** For two distributions $\mathcal{P}$, $\mathcal{Q}$ of common support, we have:

$$R_a(\mathcal{P} || \mathcal{Q}) \leq \left(1 + \frac{a(a-1)(e^{\Delta_{\mathrm{ML}}(\mathcal{P},\mathcal{Q})} - 1)^2}{2(2 - e^{\Delta_{\mathrm{ML}}(\mathcal{P},\mathcal{Q})})^{a+1}}\right)^{\frac{1}{a-1}} \underset{\Delta_{\mathrm{ML}} \to 0}{\sim} 1 + \frac{a \Delta_{\mathrm{ML}}(\mathcal{P}, \mathcal{Q})^2}{2}$$

Consequence: Instead of $\boxed{\Delta_{\mathrm{ML}} \leq 2^{-\lambda/2}}$, we only need $\boxed{\Delta_{\mathrm{ML}} \leq \frac{1}{\sqrt{\lambda q}}}$.

[1] Actually similar to the differential privacy.

Framework for using the Rényi Divergence

1. Take your favourite scheme
2. Set more aggressive parameters:
   1. First, try to apply the relative error lemma (the most powerful)
   2. Wherever it doesn't work, apply either the tailcut lemma or the reverse Pinsker's inequality

   ! Taking $R_a \leq 1 + \frac{1}{q}$ is sufficient.

   ! Taking $a \geq \lambda$ gives tight, efficient proofs.
3. Goto step 1

! *These arguments are only valid for search problems!*
For decision problems, achieving the same efficiency is still open.

! *In the rest of this presentation, we assume* $q \leq 2^{64}$.

Practice

Application 1: Security of a Sampler from [MW17]

Context. A new sampler over $\mathbb{Z}$ was introduced in [MW17].

Previous works. [MW17] perform a max-log distance-based analysis of the sampler. They find that

$$64 \text{ bits of precision} \Rightarrow \Delta_{ML} \leq 2^{-50} \Rightarrow \text{About 100 bits of security}$$

This work. We use the reverse Pinsker's inequality:

$$\begin{aligned} 64 \text{ bits of precision} \quad &\Rightarrow \quad \Delta_{ML} \leq 2^{-50} \\ &\Rightarrow \quad R_{\alpha} \leq 1 + 2^{-96} \\ &\Rightarrow \quad \text{256 bits of security, even with up to } 2^{94} \text{ queries} \end{aligned}$$

We gain this much security *for free*.
No knowledge about the sampler is required.

Application 2: Revisiting the Table Approach

Context. We study the use of precomputed tables for sampling discrete distributions – typically, (pseudo)Gaussians.

Previous works. Existing approaches [Pei10, PDG14, DG14] require high precision ($\geq \lambda/2$) and/or floating-point arithmetic.

This work. We propose a simple approach which requires less than 64 bits of *fixed* precision in practice.

The classical CDF-table approach

Let $\mathcal{D}$ be a distribution over $\mathbb{N}$ that we want to sample from.
We suppose we have a precomputed table of $\mathrm{CDF}_{\mathcal{D}}$ defined over $\mathbb{N}$ by:

$$\mathrm{CDF}_{\mathcal{D}}(z) = \sum_{i \leq z} \mathcal{D}(i)$$

Algorithm 1 CDF sampler

Require: A precomputed table of $\mathrm{CDF}_{\mathcal{D}}$

```
z ← 0
u ← [0, 1] uniformly
while u ≥ CDF_D(z) do
    z ← z + 1
Return z
```

Suppose we want to sample a half-Gaussian D_σ^+.

- *Statistical distance-based analysis.* We need to store about:
  - $\sigma \cdot \sqrt{2\lambda}$ values,
  - With a precision λ.
- *Rényi Divergence-based analysis.* We need to store about:
  - $\sigma \cdot \sqrt{2q}$ values,
  - With a precision λ. But we prefer/expect $\log_2(q)$ or $\log_2(q)/2$!

The CoDF sampler

Our solution. We use a “Rényi divergence-friendly“ table. This requires a different algorithm. We define the conditional density function of $\mathcal{D}$ by:

$$\mathrm{CoDF}_{\mathcal{D}}(z) = \mathcal{D}(z) / \sum_{i \geq z} \mathcal{D}(i)$$

Algorithm 2 CoDF sampler

Require: A precomputed table of $\mathrm{CoDF}_{\mathcal{D}}$
Ensure: $z \leftarrow \mathcal{D}$
  $z \leftarrow 0$
  $u \leftarrow [0, 1]$ uniformly
  while $u \geq \mathrm{CoDF}_{\mathcal{D}}(z)$ **do**
    $z \leftarrow z + 1$
    $u \leftarrow [0, 1]$ uniformly
  Return z

Suppose we want to sample a half-Gaussian D_{σ}^{+}.

- *Rényi Divergence-based analysis.* We need to store about:
  - $\sigma \cdot \sqrt{2q}$ values,
  - With a precision $\log_2(q)/2$!

Example and Conclusion

Gain in theory:

- CDF+SD approach: $\sigma \cdot \sqrt{2\lambda}$ values with precision λ
- CoDF+RD approach: $\sigma \cdot \sqrt{2q}$ values with precision $\log_2(q)/2$

A practical example: the distribution $D^+_{\mathbb{Z},0.85...}$ from [DDLL13].

- CDF+SD approach: 20 elements of 266 bits each $\Rightarrow\ \approx 5\,300$ bits.
- CoDF+RD approach: 11 elements of 53 bits each $\Rightarrow\ \approx 600$ bits.

Conclusion:

- Both in theory and practice, we gain an order of magnitude.
- Requires only standard (64 bits) fixed-point arithmetic.
- HIghly composable with other table-based techniques.

Application 4: Standard Deviation of Trapdoor Samplers

Context. Trapdoor sampling allows to sample a discrete Gaussian $D_{\Lambda(\mathbf{B}),\sigma,\mathbf{c}}$.

- Allows hash-and-sign, IBE [GPV08], standard model signatures [CHKP10, Boy10], hierarchical IBE [CHKP10, ABB10a, ABB10b], attribute-based encryption [Boy13, BGG+14] and so on.
- Current algorithms [Kle00, GPV08, Pei10, MP12, DP16] heavily rely on floating-point arithmetic.

This work. Two axes of improvement for trapdoor samplers:

1. Squeezing the standard deviation
2. Reducing the required precision

These had critical impacts for the signature scheme Falcon [PFH+17].

Our test subject: Klein's sampler

Algorithm 3 $\mathrm{Klein}_{\mathbf{L},\sigma}(\mathbf{t})$

Require: $\sigma \geq \eta_\epsilon(\mathbb{Z}^n) \cdot \|\mathbf{B}\|_{GS}$, the GSO $\mathbf{B} = \mathbf{L} \cdot \tilde{\mathbf{B}}$, values $\sigma_j = \sigma / \|\tilde{\mathbf{b}}_j\|$, a target $\mathbf{t}$

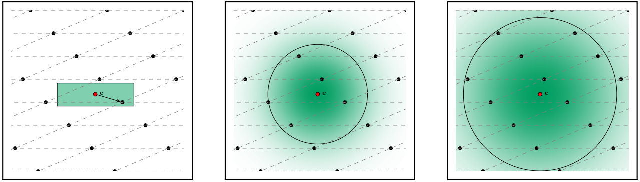
Ensure: A vector $\mathbf{z}$ such that $\mathbf{zB} \leftarrow D_{\Lambda(\mathbf{B}),\sigma,\mathbf{tB}}$

for $j = n, \ldots, 1$ **do**

$c_j \leftarrow t_j + \sum_{i>j}(t_j - z_j)L_{ij}$

$z_j \leftarrow D_{\mathbb{Z},\sigma_j,c_j}$

return z

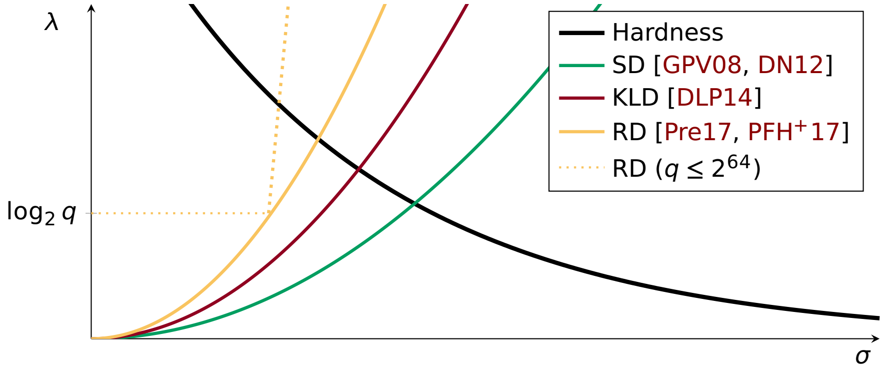


σ too small | The "right" σ | σ too large

1. σ too large $\Rightarrow$ $\mathrm{Klein}_{\mathbf{L},\sigma}$ is useless in a cryptographic context.
2. σ too small $\Rightarrow$ $\mathrm{Klein}_{\mathbf{L},\sigma}$ does not behave like a perfect Gaussian.

So σ must be small but the output of $\mathrm{Klein}_{\mathbf{L},\sigma}$ must still look like a Gaussian.

Trapdoor Sampling



The adequate value for σ is at the intersection of the hardness curve (constraint 1) and the SD/KLD/RD curve (constraint 2).

- A Rényi divergence-based analysis proves to be much more efficient than an SD/KLD-based one.
- Interesting fact: in practice, σ is not conditioned by λ but by q.

In practice, we gain about 30 bits of security (compared to the SD).

Application 5: What about the precision?

- **With the SD:** λ bits of precision
- **With the KLD [LP15]:** $\lambda/2$ bits of precision
- **With the RD [Pre17]:** $\log_2 q/2$ bits of precision

Conclusion

The current state of affairs.

- Rényi divergence is a powerful tool, but not easy to use.
- With the reverse Pinsker's inequality, the fact that the Rényi divergence is not a distance is no longer a problem.
- We can have much better parameters if these conditions are met:
  - Limited number of queries
  - Search problems
  - A bit of luck
- These results are generic (not limited to lattice-based cryptography).

Interesting questions IMHO.

- When is the Rényi divergence *worse* than the statistical distance?
- Applications outside lattice-based cryptography?
- Application to theoretical LBC rather than “production line” LBC?
- Achieve a similar efficiency for decision problems?

Interesting questions IMHO.

- When is the Rényi divergence *worse* than the statistical distance?
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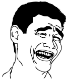
- Application to theoretical LBC rather than "production line" LBC?
- Achieve a similar efficiency for decision problems?



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